Chapter review

1 a $x = 4t - 3,\ y = \frac{8}{t^2} = 8t^{-2}$

$\frac{dx}{dt} = 4,\ \frac{dy}{dt} = -16t^{-3}$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-16t^{-3}}{4} = \frac{-4}{t^3}$$

b When $t = 2$, the curve has gradient
$\frac{dy}{dx} = \frac{-4}{2^3} = -\frac{1}{2}$

$\therefore$ the normal has gradient 2.

Also, when $t = 2$, $x = 5$ and $y = 2$,
so the point A has coordinates (5, 2).

$\therefore$ the equation of the normal at A is

$y - 2 = 2(x - 5)$

i.e. $y = 2x - 8$

2 $x = 2t,\ y = t^2$

$\frac{dx}{dt} = 2,\ \frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{2} = t$$

At the point P where $t = 3$, the gradient of the curve is $\frac{dy}{dx} = 3$

$\therefore$ gradient of the normal is $-\frac{1}{3}$

Also, when $t = 3$, the coordinates are (6, 9).

$\therefore$ the equation of the normal at P is

$y - 9 = -\frac{1}{3}(x - 6)$

i.e. $3y + x = 33$

3 $x = t^3,\ y = t^2$

$\frac{dx}{dt} = 3t^2,\ \frac{dy}{dt} = 2t$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2t}{3t^2} = \frac{2}{3t}$$

At the point (1, 1) the value of t is 1.

$\therefore$ the gradient of the curve is $\frac{2}{3}$, which is also the gradient of the tangent.

$\therefore$ the equation of the tangent is
$y - 1 = \frac{2}{3}(x - 1)$
i.e. $y = \frac{2}{3}x + \frac{1}{3}$

4 a $x = 2\cos t + \sin 2t,\ y = \cos t - 2\sin 2t$

$\frac{dx}{dt} = -2\sin t + 2\cos 2t$

$\frac{dy}{dt} = -\sin t - 4\cos 2t$

b $$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{-\sin t - 4\cos 2t}{-2\sin t + 2\cos 2t}$$

When $t = \frac{\pi}{4}$, $\frac{dy}{dx} = \frac{-\frac{1}{\sqrt{2}} - 0}{-\frac{2}{\sqrt{2}} + 0} = \frac{1}{2}$

4 c The gradient of the normal at the point P where $t=\frac{\pi}{4}$ is –2.

The coordinates of P are found by substituting $t=\frac{\pi}{4}$ into the parametric equations:

$$x=\frac{2}{\sqrt{2}}+1,\ y=\frac{1}{\sqrt{2}}-2$$

∴ the equation of the normal at P is

$$y-\left(\frac{1}{\sqrt{2}}-2\right)=-2\left(x-\left(\frac{2}{\sqrt{2}}+1\right)\right)$$

$$y-\frac{1}{\sqrt{2}}+2=-2x+2\sqrt{2}+2$$

i.e. $y+2x=\frac{5\sqrt{2}}{2}$

5 a $x=2t+3,\ y=t^3-4t$

At point A, where $t=-1$, $x=1$ and $y=3$.

∴ the coordinates of A are (1, 3).

$$\frac{dx}{dt}=2,\ \frac{dy}{dt}=3t^2-4$$

$$\therefore \frac{dy}{dx}=\frac{3t^2-4}{2}$$

At the point A, $\frac{dy}{dx}=-\frac{1}{2}$

∴ gradient of the tangent at A is $-\frac{1}{2}$

Equation of the tangent at A is

$$y-3=-\frac{1}{2}(x-1)$$

$$2y-6=-x+1$$

i.e. $2y+x=7$

b The tangent line l meets the curve C at points A and B.

Substitute $x=2t+3$ and $y=t^3-4t$ into the equation of l:

$$2(t^3-4t)+(2t+3)=7$$
$$2t^3-6t=4$$
$$t^3-3t-2=0$$

At point A, $t=-1$, so $t=-1$ is a root of this equation, and hence $(t+1)$ is a factor of the left-hand side expression.

$$t^3-3t-2=(t+1)(t^2-t-2)$$
$$=(t+1)(t+1)(t-2)$$
$$=2(t+1)^2(t-2)$$

So line l meets the curve C at $t=-1$ (repeated root because the line is tangent to the curve there) and at $t=2$.

Therefore, at point B, $t=2$.

6 The rate of change of V is $\frac{dV}{dt}$

$$\therefore \frac{dV}{dt}\propto V$$

i.e. $\frac{dV}{dt}=-kV$

where k is a positive constant.
(The negative sign is needed as the value of the car is *decreasing*.)

7 The rate of change of mass is $\frac{dM}{dt}$

$$\therefore \frac{dM}{dt}\propto M$$

i.e. $\frac{dM}{dt}=-kM$

where k is a positive constant.
(The negative sign represents *loss* of mass.)

8 The rate of change of pondweed is $\frac{dP}{dt}$

The growth rate is proportional to P:

growth rate $\propto P$

i.e. growth rate $= kP$
where k is a positive constant.

But pondweed is also being removed at a constant rate Q.

$\therefore \frac{dP}{dt} = \text{growth rate} - \text{removal rate}$

$\frac{dP}{dt} = kP - Q$

9 The rate of increase of the radius is $\frac{dr}{dt}$

$\therefore \frac{dr}{dt} \propto \frac{1}{r}$, as the rate is *inversely* proportional to the radius.

Hence $\frac{dr}{dt} = \frac{k}{r}$
where k is the constant of proportion.

10 The rate of change of temperature is $\frac{d\theta}{dt}$

$\therefore \frac{d\theta}{dt} \propto (\theta - \theta_0)$

i.e. $\frac{d\theta}{dt} = -k(\theta - \theta_0)$,
where k is a positive constant.
(The negative sign indicates that the temperature is decreasing, i.e. *loss* of temperature.)

11 a $x = 4\cos 2t,\ y = 3\sin t$

The point $A\left(2, \frac{3}{2}\right)$ is on the curve, so

$4\cos 2t = 2$ and $3\sin t = \frac{3}{2}$

$\cos 2t = \frac{1}{2}$ and $\sin t = \frac{1}{2}$

The only value of t in the interval $-\frac{\pi}{2} < t < \frac{\pi}{2}$ that satisfies both equations is $\frac{\pi}{6}$. Therefore $t = \frac{\pi}{6}$ at the point A.

b $\frac{dx}{dt} = -8\sin 2t,\quad \frac{dy}{dt} = 3\cos t$

$\therefore \frac{dy}{dx} = \frac{3\cos t}{-8\sin 2t}$

$= -\frac{3\cos t}{16\sin t\cos t}$

(using a double angle formula)

$= -\frac{3}{16\sin t}$

$= -\frac{3}{16}\operatorname{cosec} t$

c At point A, where $t = \frac{\pi}{6}$, $\frac{dy}{dx} = -\frac{3}{8}$

$\therefore$ gradient of the normal at A is $\frac{8}{3}$

Equation of the normal is

$y - \frac{3}{2} = \frac{8}{3}(x - 2)$

Multiply through by 6 and rearrange to give:

$6y - 9 = 16x - 32$

$6y - 16x + 23 = 0$

11 d To find where the normal cuts the curve, substitute $x = 4\cos 2t$ and $y = 3\sin t$ into the equation of the normal:

$$6(3\sin t) - 16(4\cos 2t) + 23 = 0$$
$$18\sin t - 64\cos 2t + 23 = 0$$
$$18\sin t - 64(1 - 2\sin^2 t) + 23 = 0$$
(using a double angle formula)
$$128\sin^2 t + 18\sin t - 41 = 0$$

But $\sin t = \frac{1}{2}$ is one solution of this equation, as point A lies on the line and on the curve. So

$$128\sin^2 t + 18\sin t - 41$$
$$= (2\sin t - 1)(64\sin t + 41)$$

$$\therefore (2\sin t - 1)(64\sin t + 41) = 0$$

Therefore, at point B, $\sin t = -\frac{41}{64}$

$\therefore$ the y-coordinate of point B is
$$3\times\left(-\frac{41}{64}\right) = -\frac{123}{64}$$

12 a $x = a\sin^2 t,\ y = a\cos t$

$$\frac{dx}{dt} = 2a\sin t\cos t,\quad \frac{dy}{dt} = -a\sin t$$

$$\therefore \frac{dy}{dx} = \frac{-a\sin t}{2a\sin t\cos t} = \frac{-1}{2\cos t} = -\frac{1}{2}\sec t$$

b As $P\left(\frac{3}{4}a, \frac{1}{2}a\right)$ lies on the curve,

$$a\sin^2 t = \frac{3}{4}a \quad\text{and}\quad a\cos t = \frac{1}{2}a$$
$$\sin t = \pm\frac{\sqrt{3}}{2} \quad\text{and}\quad \cos t = \frac{1}{2}$$

The only value of t in the interval $0 \leqslant t \leqslant \frac{\pi}{2}$ that satisfies both equations is $\frac{\pi}{3}$. Therefore $t = \frac{\pi}{3}$ at the point P.

Gradient of the curve at point P is $-\frac{1}{2}\sec\frac{\pi}{3} = -1$.

$\therefore$ equation of the tangent at P is

$$y - \frac{1}{2}a = -1\left(x - \frac{3}{4}a\right)$$
$$y - \frac{1}{2}a = -x + \frac{3}{4}a$$

Multiply equation by 4 and rearrange to give

$$4y + 4x = 5a$$

c Equation of the tangent at C is $4y + 4x = 5a$

At A, $x = 0 \Rightarrow y = \frac{5a}{4}$

At B, $y = 0 \Rightarrow x = \frac{5a}{4}$

Area of $AOB = \frac{1}{2}\left(\frac{5a}{4}\right)^2 = \frac{25}{32}a^2$,

which is of the form ka^2 with $k = \frac{25}{32}$

13 $x=(t+1)^2,\ y=\frac{1}{2}t^3+3$

$$\frac{dx}{dt}=2(t+1),\ \frac{dy}{dt}=\frac{3}{2}t^2$$

$$\therefore \frac{dy}{dx}=\frac{\frac{3}{2}t^2}{2(t+1)}=\frac{3t^2}{4(t+1)}$$

When $t=2$, $\frac{dy}{dx}=\frac{3\times4}{4\times3}=1$

$\therefore$ gradient of the normal at the point P, where $t=2$, is -1.

The coordinates of P are (9, 7).

$\therefore$ equation of the normal is

$$y-7=-1(x-9)$$
$$y-7=-x+9$$
i.e. $y+x=16$

14 $5x^2+5y^2-6xy=13$

Differentiate with respect to x:

$$10x+10y\frac{dy}{dx}-6\left(x\frac{dy}{dx}+y\right)=0$$

$$(10y-6x)\frac{dy}{dx}=6y-10x$$
$$\therefore \frac{dy}{dx}=\frac{6y-10x}{10y-6x}$$

At the point (1, 2)

$$\frac{dy}{dx}=\frac{12-10}{20-6}=\frac{2}{14}=\frac{1}{7}$$

So the gradient of the curve at (1, 2) is $\frac{1}{7}$

15 $e^{2x}+e^{2y}=xy$

Differentiate with respect to x:

$$2e^{2x}+2e^{2y}\frac{dy}{dx}=x\frac{dy}{dx}+y\times1$$

$$2e^{2y}\frac{dy}{dx}-x\frac{dy}{dx}=y-2e^{2x}$$

$$(2e^{2y}-x)\frac{dy}{dx}=y-2e^{2x}$$

$$\therefore \frac{dy}{dx}=\frac{y-2e^{2x}}{2e^{2y}-x}$$

16 $y^3+3xy^2-x^3=3$

Differentiate with respect to x:

$$3y^2\frac{dy}{dx}+\left(3x\times2y\frac{dy}{dx}+y^2\times3\right)-3x^2=0$$

$$(3y^2+6xy)\frac{dy}{dx}=3x^2-3y^2$$

$$\therefore \frac{dy}{dx}=\frac{3(x^2-y^2)}{3y(y+2x)}=\frac{x^2-y^2}{y(y+2x)}$$

Turning points occur when $\frac{dy}{dx}=0$

$$\frac{x^2-y^2}{y(y+2x)}=0$$
$$x^2=y^2$$
$$x=\pm y$$

When $x=y$, $y^3+3y^3-y^3=3$
so $3y^3=3$
$y=1$ and hence $x=1$

When $x=-y$, $y^3-3y^3+y^3=3$
so $-y^3=3$
$y=\sqrt[3]{-3}$ and hence $x=-\sqrt[3]{-3}$

$\therefore$ the coordinates of the turning points are (1, 1) and $(-\sqrt[3]{-3}, \sqrt[3]{-3})$.

17 a $(1+x)(2+y)=x^2+y^2$

Differentiate with respect to x:

$$(1+x)\left(\frac{dy}{dx}\right)+(2+y)(1)=2x+2y\frac{dy}{dx}$$

$$(1+x-2y)\frac{dy}{dx}=2x-y-2$$

$$\therefore \frac{dy}{dx}=\frac{2x-y-2}{1+x-2y}$$

b When the curve meets the y-axis, $x = 0$.

Substitute $x = 0$ into the equation of the curve:

$$2+y=y^2$$

i.e. $y^2-y-2=0$

$(y-2)(y+1)=0$

$y=2$ or $y=-1$

At $(0, 2)$, $\frac{dy}{dx}=\frac{0-2-2}{1+0-4}=\frac{4}{3}$

At $(0, -1)$, $\frac{dy}{dx}=\frac{0+1-2}{1+0+2}=-\frac{1}{3}$

c A tangent that is parallel to the y-axis has infinite gradient.

For $\frac{dy}{dx}=\frac{2x-y-2}{1+x-2y}$ to be infinite,

the denominator $1+x-2y=0$,

i.e. $x=2y-1$

Substitute $x=2y-1$ into the equation of the curve:

$(1+2y-1)(2+y)=(2y-1)^2+y^2$

$2y^2+4y=4y^2-4y+1+y^2$

$3y^2-8y+1=0$

$$y=\frac{8\pm\sqrt{64-12}}{6}=\frac{4\pm\sqrt{13}}{3}$$

When $y-\frac{4+\sqrt{13}}{3}$, $x=\frac{5+2\sqrt{13}}{3}$

When $y=\frac{4-\sqrt{13}}{3}$, $x=\frac{5-2\sqrt{13}}{3}$

$\therefore$ there are two points at which the tangents are parallel to the y-axis.

They are $\left(\frac{5+2\sqrt{13}}{3}, \frac{4+\sqrt{13}}{3}\right)$ and $\left(\frac{5-2\sqrt{13}}{3}, \frac{4-\sqrt{13}}{3}\right)$.

18 $7x^2+48xy-7y^2+75=0$

Implicit differentiation with respect to x gives

$$14x+48\left(x\frac{dy}{dx}+y\right)-14y\frac{dy}{dx}=0$$

$$(48x-14y)\frac{dy}{dx}=-14x-48y$$

$$\therefore \frac{dy}{dx}=\frac{-14x-48y}{48x-14y}=\frac{7x+24y}{7y-24x}$$

When $\frac{dy}{dx}=\frac{2}{11}$,

$$\frac{7x+24y}{7y-24x}=\frac{2}{11}$$

$$14y-48x=77x+264y$$

$$125x+250y=0$$

$$\therefore x+2y=0$$

So the coordinates of the points at which the gradient is $\frac{2}{11}$ satisfy $x+2y=0$, which means that the points lie on the line $x+2y=0$.

19 $y=x^x$

Take natural logs of both sides:

$$\ln y=\ln x^x$$

$\ln y = x\ln x$ (using properties of logarithms)

Differentiate with respect to x:

$$\frac{1}{y}\frac{dy}{dx}=x\times\frac{1}{x}+\ln x\times 1$$

$$=1+\ln x$$

$$\therefore \frac{dy}{dx}=y(1+\ln x)$$

But $y=x^x$

$$\therefore \frac{dy}{dx}=x^x(1+\ln x)$$

20 a $a^x=e^{kx}$

Take natural logs of both sides:

$$\ln a^x=\ln e^{kx}$$

$$x\ln a=kx$$

As this is true for all values of x, $k=\ln a$.

b Taking $a=2$,

$y=2^x=e^{kx}$ where $k=\ln 2$

$$\frac{dy}{dx}=ke^{kx}=(\ln 2)e^{(\ln 2)x}=2^x\ln 2$$

c At the point (2, 4), $x=2$.

$\therefore$ gradient of the curve at (2, 4) is

$$\frac{dy}{dx}=2^2\ln 2=4\ln 2=\ln 2^4=\ln 16$$

21 a $P=P_0(1.09)^t$

Take natural logs of both sides:

$$\ln P=\ln\left(P_0(1.09)^t\right)$$

$$=\ln P_0+\ln(1.09)^t$$

$$=\ln P_0+t\ln 1.09$$

$$\therefore t\ln 1.09=\ln P-\ln P_0$$

$$t=\frac{\ln P-\ln P_0}{\ln 1.09} \text{ or } \frac{\ln\left(\frac{P}{P_0}\right)}{\ln 1.09}$$

b When $t=T$, $P=2P_0$.

Substituting these into the expression in part **a** gives

$$T=\frac{\ln 2}{\ln 1.09}=8.04 \text{ (3 s.f.)}$$

21 c $\frac{dP}{dt} = P_0(1.09)^t(\ln 1.09)$

When $t = T$, $P = 2P_0$ so $(1.09)^T = 2$

Hence $\frac{dP}{dt} = P_0(1.09)^T(\ln 1.09)$

$= P_0 \times 2 \times \ln 1.09$

$= 0.172P_0$ (3 s.f.)

22 a $y = \ln(\sin x)$

$\frac{dy}{dx} = \cos x \times \frac{1}{\sin x} = \cot x$

At a stationary point $\frac{dy}{dx} = 0$

$\cot x = 0 \Rightarrow x = \frac{\pi}{2}$

(in the interval $0 < x < \pi$)

When $x = \frac{\pi}{2}$, $y = \ln\left(\sin\frac{\pi}{2}\right) = \ln 1 = 0$

$\therefore$ stationary point is at $\left(\frac{\pi}{2}, 0\right)$.

b $\frac{d^2y}{dx^2} = -\text{cosec}^2 x$

$\text{cosec}^2 x = \frac{1}{\sin^2 x} > 0$ for all $0 < x < \pi$

$\therefore \frac{d^2y}{dx^2} < 0$ for all $0 < x < \pi$

Hence the curve C is concave for all values of x in its domain.

23 a $m = 40e^{-0.244t}$

After 9 months, $t = 0.75$, so

$m = 40e^{-0.244\times0.75} = 40e^{-0.183} = 33.31...$

b $\frac{dm}{dt} = -0.244 \times 40e^{-0.244t} = -9.76e^{-0.244t}$

c The negative sign indicates that the mass is decreasing.

24 a $f(x) = \frac{\cos 2x}{e^x}$

$f'(x) = \frac{-2e^x \sin 2x - e^x \cos 2x}{e^{2x}}$

$= -\frac{2\sin 2x + \cos 2x}{e^x}$

At A and B, $f'(x) = 0$

$2\sin 2x + \cos 2x = 0$

$2\tan 2x + 1 = 0$

$\tan 2x = -0.5$

$2x = 2.678$ or 5.820

$x = 1.339$ or 2.910

(in the interval $0 \leqslant x \leqslant \pi$)

$x = 1.339 \Rightarrow y = f(x) = -0.2344$

$x = 2.910 \Rightarrow y = f(x) = 0.04874$

Therefore, to 3 significant figures:
coordinates of A are $(1.34, -0.234)$;
coordinates of B are $(2.91, 0.0487)$.

b The curve of $y = 2 + 4f(x-4)$ is a transformation of $f(x)$, obtained via a translation of 4 units to the right, a stretch by a factor of 4 in the y-direction, and then a translation of 2 units upwards.

Turning points are:
minimum $(1.34 + 4, -0.234 \times 4 + 2)$ and
maximum $(2.91 + 4, 0.0487 \times 4 + 2)$,
i.e. minimum $(5.34, 1.06)$
and maximum $(6.91, 2.19)$.

c $f''(x)$

$= -\frac{e^x(4\cos 2x - 2\sin 2x) - e^x(2\sin 2x + \cos 2x)}{e^{2x}}$

$= \frac{4\sin 2x - 3\cos 2x}{e^x}$

$f(x)$ is concave when $f''(x) \leqslant 0$

$f''(x) = 0$ when

$4\sin 2x - 3\cos 2x = 0$

$\tan 2x = \frac{3}{4}$

$2x = 0.644$ or 3.785

$x = 0.322$ or 1.893

The curve has a minimum point and hence is convex between these values, so it is concave for
$0 \leqslant x \leqslant 0.322$ and $1.892 \leqslant x \leqslant \pi$.

Challenge

a $y = 2\sin 2t,\ x = 5\cos\left(t + \frac{\pi}{12}\right)$

$\frac{dy}{dt} = 4\cos 2t,\ \frac{dx}{dt} = -5\sin\left(t + \frac{\pi}{12}\right)$

$\therefore \frac{dy}{dx} = -\frac{4\cos 2t}{5\sin\left(t + \frac{\pi}{12}\right)}$

b $\frac{dy}{dx} = 0$ when $4\cos 2t = 0$

$2t = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}$ or $\frac{7\pi}{2}$

$t = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}$ or $\frac{7\pi}{4}$

(in the interval $0 \leqslant x \leqslant 2\pi$)

$t = \frac{\pi}{4} \Rightarrow x = 5\cos\left(\frac{\pi}{3}\right) = \frac{5}{2}$

and $y = 2\sin\frac{\pi}{2} = 2$, i.e. point $\left(\frac{5}{2}, 2\right)$

$t = \frac{3\pi}{4} \Rightarrow x = 5\cos\left(\frac{5\pi}{6}\right) = -\frac{5\sqrt{3}}{2}$

and $y = 2\sin\frac{3\pi}{2} = -2$, i.e. point $\left(-\frac{5\sqrt{3}}{2}, -2\right)$

$t = \frac{5\pi}{4} \Rightarrow x = 5\cos\left(\frac{4\pi}{3}\right) = -\frac{5}{2}$

and $y = 2\sin\frac{5\pi}{2} = 2$, i.e. point $\left(-\frac{5}{2}, 2\right)$

$t = \frac{7\pi}{4} \Rightarrow x = 5\cos\left(\frac{11\pi}{6}\right) = \frac{5\sqrt{3}}{2}$

and $y = 2\sin\frac{7\pi}{2} = -2$, i.e. point $\left(\frac{5\sqrt{3}}{2}, -2\right)$

c The curve cuts the x-axis when $y = 0$,
i.e. when $2\sin 2t = 0$

$2t = 0, \pi, 2\pi, 3\pi, 4\pi$

$t = 0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$

$t = 0 \Rightarrow x = 5\cos\frac{\pi}{12} = 4.83$, i.e. $(4.83, 0)$

with gradient $\frac{dy}{dx} = \frac{-4}{5\sin\frac{\pi}{12}} = -3.09$

$t = \frac{\pi}{2} \Rightarrow x = 5\cos\frac{7\pi}{12} = -1.29$, i.e. $(-1.29, 0)$

with gradient $\frac{dy}{dx} = \frac{4}{5\sin\frac{7\pi}{12}} = 0.828$

$t = \pi \Rightarrow x = 5\cos\frac{13\pi}{12} = -4.83$, i.e. $(-4.83, 0)$

with gradient $\frac{dy}{dx} = \frac{-4}{5\sin\frac{13\pi}{12}} = 3.09$

$t = \frac{3\pi}{2} \Rightarrow x = 5\cos\frac{19\pi}{12} = 1.29$, i.e. $(1.29, 0)$

with gradient $\frac{dy}{dx} = \frac{4}{5\sin\frac{19\pi}{12}} = -0.828$

The curve cuts the y-axis when $x = 0$.

i.e. when $5\cos\left(t + \frac{\pi}{12}\right) = 0$

$t + \frac{\pi}{12} = \frac{\pi}{2}, \frac{3\pi}{2}$

$t = \frac{5\pi}{12}, \frac{17\pi}{12}$

$t = \frac{5\pi}{12} \Rightarrow y = 2\sin\frac{5\pi}{6} = 1$, i.e. $(0, 1)$

with gradient $\frac{dy}{dx} = \frac{-4\cos\frac{5\pi}{6}}{5\sin\frac{\pi}{2}} = 0.693$

$t = \frac{17\pi}{12} \Rightarrow y = 2\sin\frac{17\pi}{6} = 1$, i.e. $(0, 1)$

with gradient $\frac{dy}{dx} = \frac{-4\cos\frac{17\pi}{6}}{5\sin\frac{3\pi}{2}} = -0.693$

So the curve cuts the y-axis twice at (0, 1) with gradients 0.693 and −0.693.

Challenge

d

$$\frac{dx}{dy} = -\frac{5\sin\left(t + \frac{\pi}{12}\right)}{4\cos 2t}$$

$$\frac{dx}{dy} = 0 \text{ when } \sin\left(t + \frac{\pi}{12}\right) = 0$$

$$t + \frac{\pi}{12} = \pi, 2\pi$$

$$t = \frac{11\pi}{12}, \frac{23\pi}{12}$$

$$t = \frac{11\pi}{12} \Rightarrow y = 2\sin\frac{11\pi}{6} = -1$$

$$\text{and } x = 5\cos\left(\frac{11\pi}{12} + \frac{\pi}{12}\right) = -5$$

$$t = \frac{23\pi}{12} \Rightarrow y = 2\sin\frac{23\pi}{6} = -1$$

$$\text{and } x = 5\cos\left(\frac{23\pi}{12} + \frac{\pi}{12}\right) = 5$$

So points where curve is vertical are
(−5, −1) and (5, −1).

e